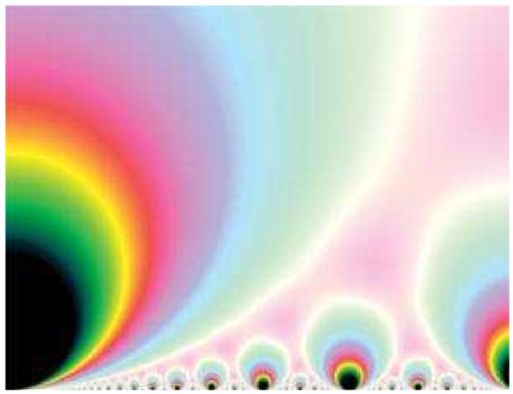
Wonder walls: Taking home decor to another dimension

16 December 2010 by **Stephen Ornes**
Magazine issue 2790. **Subscribe and save**

Having trouble decorating your hyperbolic home? Fire your interior designer and call in a mathematician



Exotic geometry (Image: S. Frank/A. Farris)

1 more image

THE most striking feature of Frank Farris's house is a stunning stained-glass window, backlit by the bright blue skies of San José, California. To the untrained eye, the pattern is just a colourful sun-catcher made up of overlapping circles, but to Farris it is as close as we can get to looking at a five-dimensional cube.

More traditional designs, grounded in the laws of geometry you learned at school, are just plain dull for Farris, who is a mathematician at nearby Santa Clara University.

Take wallpaper, for example. To mathematicians, almost all wallpaper designs can be defined with some very simple rules: they are infinitely repeating patterns that look the same when we either reflect them across certain axes, or perform a translation (a horizontal or vertical shift), or rotate them by 60, 90, 120 or 180 degrees - the only rotations that can lead to infinite repetitions. We can also carry out some combination of these transformations. In total, there are only 17 possible types of pattern that obey these rules (see "Wallpaper symmetries").

This 17-pattern limit only holds, however, if you follow the familiar rules of geometry that apply on a flat plane. For a geometer like Farris, with a penchant for outré decoration, it is more interesting to look at surfaces that break these laws - weird spaces where the size of an object depends on its position and it is quickest to walk in a curved path rather than a straight line. So, bored with the constraints of conventional design, he decided to investigate wallpapers to suit these exotic spaces.

Such thought experiments are common in the history of mathematics. Perhaps the most famous example is explored in the novel *Flatland by* Edwin A. Abbott. Published in 1884, it considers the quirks of living in a completely flat, two-dimensional universe. The story helped outline the problems mathematicians face when they consider spaces with more than three dimensions, and has become something of a cult classic. Sci-fi writer Isaac Asimov was reportedly a big fan, and the book has inspired numerous spin-offs, including an animated feature film voiced by Martin Sheen.

Flatland is still pretty conventional fare, however, since it adheres to the rules of geometry laid down by the mathematician Euclid around 2300 years ago in ancient Greece. He came up with five postulates behind all the conventional geometric principles you might have learned at school, such as the Pythagorean theorem. The first four postulates are pretty intuitive: postulate one, for example, states that you can always draw a line connecting two points across the shortest distance; postulate four states that all right angles are congruent.

Breaking the rules

The fifth postulate leads to the most intriguing consequences. In one form, it states that if you have an infinite straight line L, and a point P that is not on L, there will always be just one infinite line that passes through P and does not intersect L - a line parallel to L, in other words. By the early 19th century, mathematicians had realised that you could do away with this postulate and develop whole new kinds of geometry based solely on the first four postulates. The spaces where these geometries apply have many geometric rules that are consistent within themselves but which are surprisingly different from the Euclidean laws of *Flatland*, where the parallel postulate still holds true.

The sphere is one such non-Euclidean geometry. When he set out to devise new forms of wallpaper, however, Farris settled on something even more exotic - the Poincaré upper half-plane.

This surface looks like an infinite sheet cut in half, so it has a single edge. "Poincarites" living in this half-plane would never know about the edge of this world, though, thanks to a peculiar consequence of its "hyperbolic geometry". "They cannot see it because they and all their matter shrink as they approach the edge," Farris told an audience at MathFest 2010 in Pittsburgh, Pennsylvania, last August.

This means that any vehicles travelling towards the edge would get smaller and smaller and cover less and less distance as their journey progressed, so they could never quite reach it. The inhabitants would be oblivious to their changing size, however, since their roads and rulers would also shrink as they moved towards the edge.

This simple law subverts basic assumptions about space, distance and shapes. For one thing, it means that there is room to cram an infinite number of objects between a given point and the half-plane's boundary. As Farris puts it, "There's too much space in this space."

Things get stranger still if you consider the shortest path between two points. These lines are known as geodesics, and on a Euclidean flat surface they are simply straight lines. On the Poincaré half-plane, geodesics are only straight lines if the points in question are arranged "vertically", so the geodesic is perpendicular to the edge. But if the points in question are arranged "horizontally" in the same direction as the plane's edge, the geodesic is curved. And if three geodesics join to form a triangle in this space, the internal angles will always add up to less than 180 degrees - the total they must add up to in Euclidean space.

> In these weird spaces, an object's size depends on its position and the shortest distance between two points is a curve

Defining a parallel line is difficult in a geometry where geodesics are curved. In this case, two geodesics are considered parallel if they would never intersect on the plane, even if you extended them to an infinite length. On the Poincaré half-plane, each geodesic (L) has an infinite number of parallels through any point P - thus breaking the parallel postulate, which states there must only be one.

Farris advises Poincarites to carry a map, since it's very easy to get lost in this space. If he or she were to set off on a walk in which each step is chosen at random, the probability of returning to the starting point is almost zero. Heat would also dissipate more quickly than in Euclidean space. "Hyperbolic space is cold and lonely," Farris says.

For Farris - and perhaps no one else in the world - creating wallpaper for the Poincarites allowed him to blend his passion for mathematical symmetries with musings about life in the half-plane. Besides, he was tired of conventional wallpapers, created "as if you were taking a potato and stamping out a pattern", he says.

> Farris was tired of conventional designs, created as if you were taking a potato and stamping out a pattern

So he set about finding the rules that would govern the Poincarite wallpaper patterns (or tiling, since they wouldn't have walls in this 2D space). In Flatland, the reflections, translations and rotations that leave wallpaper looking the same do so because they preserve the size and shape of the original pattern. Mathematicians call such transformations "isometries". Preserving size is a tricky business in a hyperbolic world in which rulers shrink as they travel towards the half-plane's edge, so these three transformations do not preserve wallpaper patterns in the same way on the Poincaré half-plane.

Take reflection, for example. Although a shape reflected across a vertical line maintains its size and shape, reflections across horizontal lines are more complicated. For one thing, you would need to reflect across a curved line, since geodesics are curved in this direction. To make matters worse, rulers are different sizes on opposite sides of the reflecting line, so the size and proportions of the reflected shape will have changed from the Poincarite perspective.

Inversions

Farris wasn't daunted. He found that the Poincarite equivalent of a reflection is the mathematical operation called inversion, which exchanges the inside and outside of a circle. Like a reflection, it keeps the size and form of a shape in the hyperbolic plane constant from the perspective of the inhabitants, while reversing its orientation. Working his way through rotations and translations, Farris tweaked each isometry to preserve size and shape according to the rules of the Poincarite world. Horizontal translations remain as they are in Euclidean space, while vertical translations involve stretching the plane along the vertical axis. Hyperbolic rotations are more similar to the rotations you find in Euclidean space, except they too stretch and squeeze shapes depending on which way you turn.

Having identified these isometries, Farris needed to construct wallpaper patterns using mathematical functions that would remain unchanged under different combinations of these transformations. He struck gold with certain trigonometric functions whose value depended on both the x and y coordinates of points in the plane. He then plotted each function by converting its values into colours, with beautiful results. In the image on page 46, for example, the design is left unchanged by the equivalent of a 120-degree rotation around any of the points where two semicircles meet a vertical line. The same applies to a 180-degree rotation around any of the points where a vertical line cuts through a semicircle at a right angle.

These designs have been a labour of love for Farris. Since embarking on this work in 2001, he has updated the story with presentations, talks and new visualisations. The latest developments will be found in *Mathematical Expeditions*, a forthcoming anthology to be published by the Mathematical Association of America.

Unlike the five-dimensional cube on his stained-glass window, however, his wallpaper designs have remained theoretical. "I would dearly love for a wallpaper manufacturer to print these designs for a room with wainscoting, so that the active part of the design would be at waist-level," he says.

Sadly, no one has yet taken up the idea, though the designs have found a fan in Nancy Mims, a wallpaper designer and creative director of Mod Green Pod, a print company in Austin, Texas. She thinks Farris's designs would be particularly popular with kids. They might make an interesting talking point for adults, too - perhaps in a bar "where people might want to contemplate them while listening to music or drinking".

But how do the mathematically inspired wallpapers compare with the patterns that are in vogue with interior designers? "Right now, the look in wallpaper seems to be a very modern play on traditional motifs, so this is a nice different route," she says.

Stephen Ornes is a freelance writer based in Nashville, Tennessee

From issue 2790 of New Scientist magazine, page 44-47.

Back to article